Propagator:

$$\Delta_F(x) \equiv (2\pi)^{-4} \int d^4q \, \frac{\exp(iq\cdot x)}{q^2 + m^2 - i\varepsilon}$$

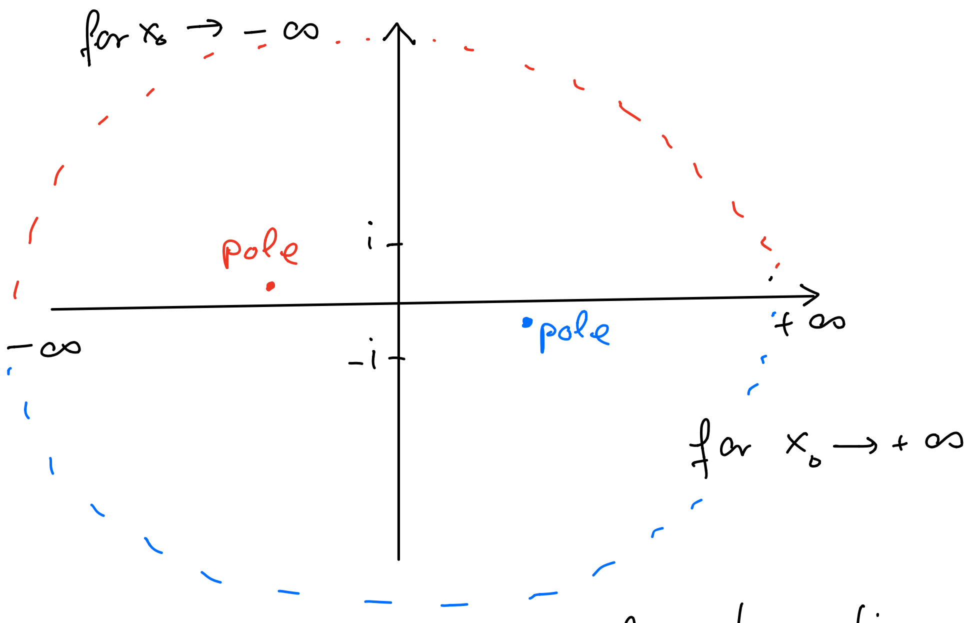
$$= (2\pi)^{-4} \int d^3q \int dq_0 \, \frac{\exp\left(i(\vec{q}\cdot\vec{x} - q_0 x_0)\right)}{\vec{q}^{\,2} - (q_0)^2 + m^2 - i\varepsilon} \quad (1)$$

$\rightarrow$ has poles at

$$q_0 = \pm\sqrt{\vec{q}^{\,2} + m^2 - i\varepsilon}$$

$$= \pm\sqrt{\vec{q}^{\,2} + m^2} \mp \frac{i\varepsilon}{2\sqrt{\vec{q}^{\,2}+m^2}} + \mathcal{O}(\varepsilon^2)$$

$\rightarrow$ is in lower/upper half plane



We close the contour of integration in the lower half-plane for $x_0 \rightarrow +\infty$ and in the upper half-plane for $x_0 \rightarrow -\infty$

Thus

$$\Delta_F(x) = (2\pi)^{-4}\left[\int d^3q \, \frac{e^{i(\vec{q}\cdot\vec{x} - ix_0\sqrt{\vec{q}^2+m^2})}}{2\sqrt{\vec{q}+m^2}}\right]\theta(x)$$

$$+ (2\pi)^{-4}\left[\int d^3q \, \frac{e^{i(\vec{q}\cdot\vec{x} + ix_0\sqrt{\vec{q}^2+m^2})}}{2\sqrt{\vec{q}+m^2}}\right]\theta(-x)$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

This can be quivalently written as

$$-i\Delta_F(x) = \theta(x)\,\Delta_+(x) + \theta(-x)\,\Delta_+(-x) \qquad (2)$$

where we have defined

$$\Delta_+(x) = (2\pi)^{-3}\int d^3p \, (2p^0)^{-1} e^{ip.x}$$

in which p^0 is taken as $+\sqrt{\vec{p}^2+m^2}$

We recognize in (1) the Green's function for the Klein-Gordon operator:

$$(\Box - m^2)\,\Delta_F(x) = -\delta^4(x)$$

Check:

$$(\Box - m^2)\,\Delta_F(x)$$

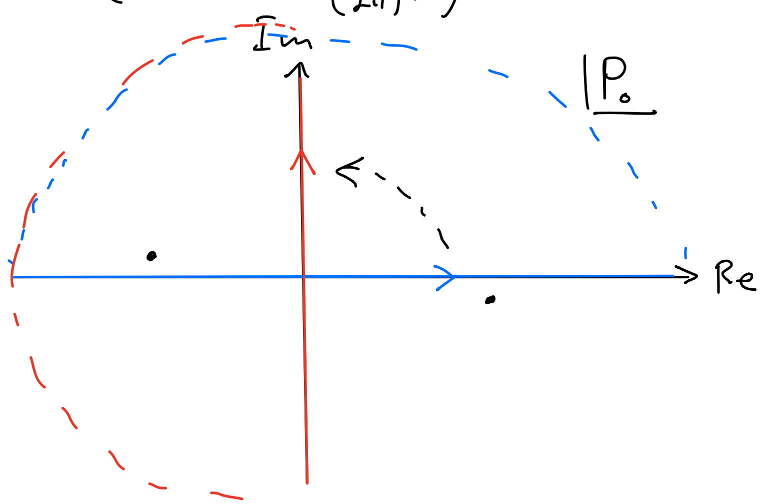
$$= (2\pi)^{-4} \int d^4q \; \frac{(\Box - m^2)\exp(iq\cdot x)}{q^2 + m^2 - i\varepsilon}$$

$$= (2\pi)^{-4} \int d^4q \; \frac{(-q^2 + m^2)\exp(iq\cdot x)}{q^2 + m^2 - i\varepsilon}$$

$$= -\,\delta^4(x)$$

Now let us come back to the expression from last lecture:

$$I(\mu^2) = -\frac{i}{(2\pi)^4} \int d^4p \left(p^2 + \mu^2 - i\varepsilon\right)^{-3}$$



$\longrightarrow$ Rotate the contour in the p_0-plane by 90 degrees counter-clockwise

§2.4 Symmetries of the Effective Action

When are symmetries of $I[\phi]$ also symmetries of $\Gamma[\phi]$?

In some cases this is automatic:

Consider the action

$$I[\phi] \equiv -\int d^4x \left[\lambda + \frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{24}g\phi^4\right]$$

$\rightarrow$ is invariant under $\phi \mapsto -\phi$

From the definition of $Z[J]$

$$Z[J] = \int \left[\prod d\phi(y)\right] \exp\left(iI[\phi] + i\int d^4x\, \phi(x) J(x)\right)$$

we see that $Z[-J] = Z[J]$

(Just redefine integration variables $\phi \rightarrow -\tilde{\phi}$)

$\rightarrow$ $W[J] = -i \log Z[J]$ is also even

Thus $\quad \phi_J \equiv \frac{\delta}{\delta J} W[J]$ is odd:

$$\phi_{-J} = -\phi_J \rightarrow J_{-\phi} = -J_\phi$$

and therefore

$$\Gamma[\phi] \equiv -\int d^4x\, \phi(x) J_\phi(x) + W[J_\phi]$$

is even under $\phi \mapsto -\phi$.

Assume now the contrary:

$I[\phi]$ is an action even under $\phi \mapsto -\phi$ but $\Gamma[\phi]$ turns out not to be

$\rightarrow$ there will be terms proportional to $\int d^4x\, \phi$ and $\int d^4x\, \phi^3$ in $\Gamma[\phi]$ with divergent coefficients (coming from loop integrations)

$\rightarrow$ the symmetries of action $I[\phi]$ does not allow us to introduce counterterms to absorb these infinities.

<u>More rigorous treatment:</u>

Consider symmetries generated by

$$\chi^n(x) \longrightarrow \chi^n(x) + \varepsilon F^n[x;\chi],$$

where F^n is a function of x^n that depends functionally on χ^n (here χ^n are meant to include not only matter fields but also ghost fields).

symmetry: $\quad I[\chi + \varepsilon F] = I[\chi]$,

$$\prod_{n,x} d\left(\chi^n(x) + \varepsilon F^n[x;\chi]\right) = \prod_{n,x} d\chi^n(x)$$

Thus

$$Z[J] = \int \left[\prod_{n,x} d\left(X^n(x) + \varepsilon F^n[x;X]\right)\right]$$

$$\times \exp\left\{ iI[X+\varepsilon F] + i\int d^4x \left(X^n(x) + \varepsilon F^n[x;X]\right) J_n(x) \right\}$$

$$= \int \left[\prod_{n,x} dX^n(x)\right] \exp\left\{ iI[X] + i\int d^4x \left(X^n(x) + \varepsilon F^n[x;X]\right) J_n(x) \right\}$$

$$= Z[J] + i\varepsilon \int \left(\prod_{n,x} dX^n(x)\right) \int F^n[y;X]\, J_n(y)\, d^4y$$

$$\times \exp\left\{ iI[X] + i\int d^4x\, X^n(x) J_n(x) \right\}$$

and hence

$$\int d^4y \, \langle F^n(y) \rangle_J \, J_n(y) = 0 ,$$

where $\langle \ \rangle_J$ denotes the quantum average in the presence of the current $J_n(x)$,

$$Z[J] \langle F^n(y) \rangle_J \equiv \int \left(\prod_{n,x} dX^n(x)\right) F^n(y;X)$$

$$\times \exp\left\{ i\, I[X] + i\int d^4x\, X^n(x) J_n(x) \right\} ,$$

normalized so that $\langle 1 \rangle_J = 1$.

Using $\quad J_{n,X}(y) = - \dfrac{\delta \Gamma[X]}{\delta X^n(y)}$

$$\longrightarrow \quad 0 = \int d^4y \, \langle F^n(y) \rangle_{J_X} \frac{\delta\Gamma[X]}{\delta X^n(y)}$$

In other words, $\Gamma[X]$ is invariant under

$$X^n(y) \longrightarrow X^n(y) + \varepsilon \langle F^n(y) \rangle_{J_X} . \qquad (*)$$

$\longrightarrow$ "Slavnov-Taylor identities"

For "linear" transformations

$$F^n[x; X] = s^n(x) + \int t^n{}_m(x,y) X^m(y)\, d^4y$$

$$\longrightarrow \langle F^n(x) \rangle_J = s^n(x) + \int t^n{}_m(x,y) \langle X^m(y) \rangle_J\, d^4y$$

$$= s^n(x) + \int t^n{}_m(x,y)\, X^m(y)\, d^4y$$

$$= F^n[x; X] .$$

Thus equation $(*)$ tells us that $\Gamma[X]$ is invariant under all functional linear trfs. that leave $I[X]$ invariant.

This is not true for non-linear trfs. !